II. *On the Volumes of Pedal Surfaces.* By T. A. Hirst, *F.R.S.*

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1. In accordance with the proposition recently made by Dr. Salmon in his excellent treatise on Surfaces*, the term *pedal surface* is here adopted, as the English equivalent of the French *surface-podaire* and the German *Fusspuncts-Fläche*, to indicate the locus of the feet of perpendiculars, let fall from one and the same point in space, upon all the tangent planes of a given *primitive surface*.

The point of contact of the tangent plane, and the foot of the perpendicular upon the latter, are said to be *corresponding points* on the primitive and its pedal. The point whence perpendiculars are let fall may be termed the *pedal-origin*. It is obvious that the pedal surface may likewise be regarded as the envelope of spheres having for their diameters the radii vectores from this origin to the several points of the primitive†.

The primitive surface remaining unaltered, the magnitude and form of the pedal will of course vary with the position of its origin. Between the volumes of all such pedals, however, certain very general and remarkable relations exist. The object of the present paper is to establish these relations.

2. Twenty-four years ago‡ Professor Steiner, in one of his able and purely geometrical memoirs presented to the Academy of Berlin, established analogous relations between the areas of *pedal curves* corresponding to different origins in the plane of the primitive. I am not aware, however, of any attempt having been made to extend his results to surfaces, although such an extension can scarcely have failed to suggest itself, not only to Steiner himself, but to many of his readers§. For the sake of comparison I will here state a few of these results.

* A Treatise on the Analytic Geometry of Three Dimensions, by G. Salmon, D.D., 1862, p. 369.

† The pedal origin being the same, the surface derived from the pedal, in the same manner as it was derived from the primitive, would be called the *second pedal*; the pedal of this, again, the *third pedal*, and so on. It has, further, been found convenient to apply the term *positive* to the pedals of this series, in order to distinguish them from another series of surfaces obtained by reversing the above process of derivation. Thus the surface of which the primitive is the pedal is termed the *first negative pedal*, and so on. I may also remark that the whole series of positive and negative pedals is identical with the series of *derived surfaces* which forms the subject of papers published by Messrs. Tortolini and W. Roberts, as well as by myself, in Tortolini's 'Annali' and the 'Quarterly Journal of Math.' for 1859. In the present paper first positive pedals are alone considered, though it would no doubt be interesting to examine the volumes of pedals of higher order.

‡ See Crelle's Journal, vol. xxi. p. 57.

§ Dr. Borchardt has quite recently (April 1863) apprized me of the existence of an Inaugural Dissertation, entitled "De superficierum pedalium theorematibus quibusdam," whose publication was sanctioned, in 1859, by the University of Berlin, and in which the two fundamental theorems of art. 3 are established. To English mathematicians, however, the theorems in question will probably be still new, since, so far as I can ascertain, their discoverer, Dr. Fischer, has never given full publicity to the results of his investigations.

"The primitive curve being closed, but otherwise perfectly arbitrary, the locus of the origins of pedals of constant area is a circle. The several circular loci, corresponding to different areas, are concentric, and their common centre is the origin of the pedal of minimum area."

STEINER signalizes, as a very remarkable mechanical property of this common centre, the fact that it always coincides with the *Krümmungs-Schwerpunct* of the primitive curve,—that is to say, with the centre of gravity of that primitive, regarded as a material curve whose density is everywhere proportional to the curvature.

In 1854, sixteen years after the appearance of STEINER'S memoir, Professor RAABE of Zürich* extended STEINER'S theorem so as to embrace the pedals of unclosed curves. The general definition of the area of a pedal being the space swept by the perpendicular as the point of contact of the tangent describes the primitive arc, RAABE found that "the origins of all pedals of the same area lie on a conic." The several quadric loci, corresponding to different areas, are concentric and co-axal; their common centre is again the origin of the pedal of least area; and though it no longer coincides with the *Krümmungs-Schwerpunct* of the primitive arc, it is intimately connected therewith, as has been more recently shown by Dr. WETZIG of Leipzig†.

3. With respect to surfaces, the volume of the pedal may be stated, in general terms, to be that of the cone whose vertex is the pedal-origin and whose base is that portion of the pedal surface which corresponds to the given portion of the primitive. This definition being accepted, it will be shown in the sequel that, *whatever the nature of the primitive surface may be, the origins of pedals of equal volume always lie on a surface of the third order;* and further, that *when the primitive surface is closed, but otherwise perfectly arbitrary, this cubic locus degenerates to a quadric, the whole of the loci, corresponding to all possible volumes, then forming a system of similar, similarly placed, and concentric quadrics whose common centre is the origin of the pedal of least volume.*

4. For the sake of comparison it is desirable to treat, by a uniform method, the two analogous questions respecting pedal curves and pedal surfaces. I commence, therefore, with a brief consideration of STEINER'S theorem.

Let (C) represent the primitive curve, (P) the pedal whose origin A has the coordinates x, y, and (P_0) the pedal whose origin O coincides with that of the coordinate axes. The curve (C) may be regarded as dividing the plane into two parts, distinguishable as external and internal; let α and β then be the angles, each positive and less than π, between the positive directions of the coordinate axes and that of the normal at any point M of (C), this normal being always supposed to be drawn from the curve into the external part of the plane. Further, let p and p_0 be the perpendiculars let fall respectively from the point A, and from the origin O upon the tangent at M, so that their feet m and m_0 are the points on the pedals (P) and (P_0) which correspond to M on the primitive. The direction-angles of each perpendicular will be

$$\alpha,\ \beta,\ \text{or}\ \pi-\alpha,\ \pi-\beta,$$

* CRELLE'S Journal, vol. l. p. 193.

† Zeitschrift für Mathematik und Physik, 1860, vol. v. p. 81.

according as its direction coincides with, or is opposed to that of the normal; so that if we regard p and p_0 as positive or negative according as the one or the other of these circumstances occurs, we shall have, generally,

$$p=p_0-x\cos\alpha-y\sin\alpha.$$

If we, further, denote by $d\theta$ the arc of the unit-circle, around the origin, intercepted between radii whose directions coincide with those of the normals at the extremities of the element ds of the primitive arc at M, and agree to consider the parallel elements ds and $d\theta$ as alike or unlike in sign according as their directions coincide with or are opposed to each other, the corresponding elements dP and $d\mathrm{P}_0$ of the areas of the pedals (P) and (P_0) will be

$$d\mathrm{P}=\frac{p^2d\theta}{2},\ d\mathrm{P}_0=\frac{p_0^2d\theta}{2},$$

and, by the preceding relation, we shall have

$$2d\mathrm{P}=(p_0-x\cos\alpha-y\sin\alpha)^2d\theta;$$

whence, by integration, we deduce the equation

$$\mathrm{P}=\mathrm{P}_0-\mathrm{A}_1x-\mathrm{A}_2y+\tfrac{1}{2}(\mathrm{A}_{11}x^2+2\mathrm{A}_{12}xy+\mathrm{A}_{22}y^2), \quad \text{(A.)}$$

wherein P and P_0 denote the areas of the two pedals, and the coefficients have the values

$$\mathrm{A}_1=\int p_0d\theta.\cos\alpha,\quad \mathrm{A}_2=\int p_0d\theta.\cos\beta,$$

$$\mathrm{A}_{11}=\int d\theta\cos^2\alpha,\quad \mathrm{A}_{12}=\int d\theta\cos\alpha\cos\beta,\quad \mathrm{A}_{22}=\int d\theta\cos^2\beta,$$

dependent only on the position of the origin O, and on the curvature of the primitive curve. The integration in each case is, of course, to be extended to all points of the primitive arc.

5. The above formula, by means of which the area of any pedal (P) may be found when the area of any other (P_0) is known, shows at once that the locus (A) of the origin A of a pedal (P) of constant area is a conic, and that all such loci constitute a system of similar, similarly placed, and concentric conics, the common centre of the loci being the point at which the integrals A_1, A_2 vanish. If we suppose the origin of our coordinate axes to coincide with this point, the equation of the locus (A) may be written thus:

$$\mathrm{P}=\mathrm{P}_0+\tfrac{1}{2}\int(x\cos\alpha+y\cos\beta)^2d\theta,$$

whence we learn that the common centre of all the quadric loci (A) is the origin of the pedal of least area.

6. This is RAABE's theorem; in order to deduce STEINER's from it let us consider, in the first place, the pedals of a primitive arc containing a point of inflexion and having parallel normals at its extremities. The normals along such an arc will consist of pairs of like-directed parallels; but in passing from one extremity to the other the sign

of $d\theta$ will change, so that the integrals A_{11}, A_{12}, A_{22} will each consist of equal and opposite elements and vanish in consequence*.

If, now, the primitive be a closed curve, but otherwise perfectly arbitrary, we may always conceive it to consist of arcs (C′) of the kind just considered, and of other arcs (C″) the directions of whose normals represent exactly all possible directions round a point. But it has already been shown that for every arc (C′) the integrals A_{11}, A_{12}, A_{22} vanish, and it is easy to see that, extended over the arcs (C″), these integrals have the values

$$A_{11}=A_{22}=n\pi,\ A_{12}=0,$$

where n represents the number of such arcs, in other words, the number of convolutions of the primitive curve. In this case, therefore, the equation of art. 5 becomes

$$P=P_0+\frac{n\pi}{2}(x^2+y^2)=P_0+\frac{n\pi}{2}r^2,$$

and for constant values of P represents a circle around the origin of the least pedal.

7. In order to illustrate by an example what is meant by the area of a pedal, let us consider for a moment the case of an ellipse with the semiaxes a, b. The focal pedal, as is well known, is a circle whose diameter is the major axis; so that putting for P, n, r^2 the values πa^2, 1, a^2-b^2 respectively, we find, for the area of the central pedal, the value

$$P_0=\frac{\pi}{2}(a^2+b^2),$$

equal to the area of the semicircle whose radius is the line joining the extremities of the axes; and the area of any other pedal is

$$P=\frac{\pi}{2}(a^2+b^2+r^2).$$

For the circle $a=b$, we have

$$P=\pi a^2+\frac{\pi}{2}r^2,$$

which clearly represents the *sum* of the areas of the two loops of which the pedal consists when its origin is without the circle. When a vanishes, the pedal is well known to be the circle on r as diameter. Our last formula shows, however, that we must conceive this circle to be doubled. A glance at the expressions for p and dP in art. 4 explains this distinctive feature of pedal areas. It will be there seen that the sign of the increment dP does not depend upon that of p, which latter changes according as the pedal-origin lies on one or the other side of the tangent. For pedal surfaces, to which we will now proceed, the case is otherwise.

8. Let x, y, z be the coordinates of the origin A of a pedal (P) of a surface (S); and, as before, let (P_0) denote the pedal of the same surface whose origin O coincides with that of the coordinate axes. Then if α, β, γ be the direction-angles of the *external*

* The locus (A) of equal pedal origins coincides, in this case, with the right line $P=P_0-A_1x-A_2y$, as was first shown by WETZIG.

normal at a point M of (S), and p, p_0 the perpendiculars from the origins of (P) and (P_0) upon the tangent plane at M, we shall, again, have the general relation

$$p = p_0 - x\cos\alpha - y\cos\beta - z\cos\gamma,$$

provided the sign of p be understood to depend upon the side of the tangent plane upon which the pedal origin is situated.

Further, let $d\sigma$ be the surface-element of the unit-sphere intercepted by radii having precisely the same directions as the external normals at the contour of the element ds at M on the primitive surface. According to GAUSS's definition $d\sigma$ will also be the *total curvature* of the element ds, and will have the value kds, where k is the *measure of curvature* at M, in other words, the reciprocal of the product of the principal radii of curvature. The volume-element of the pedal P will, obviously, have the value

$$dP = \tfrac{1}{3}p^3 d\sigma,$$

and will change sign with p as well as with $d\sigma$. By means of the preceding relation, then, we have

$$3dP = (p - x\cos\alpha - y\cos\beta - z\cos\gamma)^3 d\sigma,$$

which expression, when developed and integrated, assumes the form

$$\left.\begin{array}{l} P = P_0 - (A_1, A_2, A_3 \between x, y, z) + (A_{11}, A_{22}, A_{33}, A_{23}, A_{31}, A_{12} \between x, y, z)^2 \\ -\tfrac{1}{3}(A_{111}, A_{222}, A_{333}, A_{112}, A_{113}, A_{223}, A_{221}, A_{331}, A_{332}, A_{123} \between x, y, z)^3, \end{array}\right\} \quad . \quad . \quad . \quad \text{(A.)}$$

where the nineteen coefficients are independent of the position of the pedal origin A, and represent double integrals to be extended to all points of the primitive surface. Of these coefficients it will suffice to write the values of the following six, the remaining thirteen being deducible therefrom by permutations of α, β, γ in accordance with those of the suffixes 1, 2, 3.

$$A_1 = \int p_0^2 d\sigma\cos\alpha \quad , \quad A_{11} = \int p_0 d\sigma\cos^2\alpha \quad , \quad A_{111} = \int d\sigma\cos^3\alpha,$$

$$A_{12} = \int p_0 d\sigma\cos\alpha\cos\beta, \quad A_{112} = \int d\sigma\cos^2\alpha\cos\beta, \quad A_{123} = \int d\sigma\cos\alpha\cos\beta\cos\gamma.$$

The above formula for the volume of the pedal (P) at any point A shows at once, as stated in art. 3, that *the origins of pedals of equal volume are situated on a surface of the third order*.

9. The analogy between the cases of pedal curves and surfaces will be evident on observing that the above cubic locus proceeds essentially from the three dimensions of space, just as the quadric locus, in the case of pedal curves, was due to the two dimensions of a plane. It is interesting to note, however, that whilst the hypothesis of a closed primitive *curve* had merely the effect of altering the species, not the order, of the locus (A), the hypothesis of a closed primitive *surface* leads to a reduction of this locus from a cubic to a quadric. The former effect was produced by the equalization of the coefficients of x^2 and y^2, and the vanishing of that of xy (art. 6); the latter is due to

the vanishing of each of the ten integrals A_{111}, A_{112}, &c.... which, not involving p_0, have values dependent solely upon the *curvature* of the primitive surface.

I do not attempt any complete discussion of all possible singularities of curvature, but merely observe that the above-mentioned property of the ten integrals is easily recognized when the primitive surface is not only closed, but everywhere convex; for since all directions round a point are then exactly represented by its normals, the integrals in question each represent a sum of pairs of equal and opposite elements. In the more general case, where certain directions are represented more than once, and consequently an odd number of times, by the normals of the primitive, the property in question may be verified by a method similar to that employed in art. 6.

10. The primitive being a closed surface, the form to which the equation (A.) of art. 8 becomes reduced, at once shows that *the several quadric loci corresponding to pedals* (P) *of different, but constant volumes, constitute a system of similar, similarly situated, and concentric quadrics, their common centre being the origin of the pedal of least volume.* For if this centre, which is determined by the conditions

$$A_1=\int p_0^2 d\sigma \cos\alpha=0,\quad A_2=\int p_0^2 d\sigma \cos\beta=0,\quad A_3=\int p_0^2 d\sigma \cos\gamma=0,$$

were chosen as origin of coordinate axes, the equation (A.) of art. 8 would assume the form

$$P=P_0+(A_{11}, A_{22}, A_{33}, A_{23}, A_{31}, A_{12})(x, y, z)^2,$$

which may be also written thus,

$$P=P_0+\int p_0(x\cos\alpha+y\cos\beta+z\cos\gamma)^2 d\sigma,$$

in which form it renders apparent the minimum property in question.

When the closed primitive has itself a centre, the latter will also be the common centre of the loci (A); for, the centre of the primitive being taken as origin of coordinate axes, each of the integrals A_1, A_2, A_3 will again consist of pairs of equal and opposite elements.

11. To illustrate the foregoing principles, as well as to facilitate future applications, we will consider for a moment the simplest of all cases—where the primitive is a sphere with radius a. Taking its centre for origin, sixteen of the integrals of art. 8 will be found to vanish, and the remaining ones, A_{11}, A_{22}, A_{33}, to acquire the common value $\frac{4}{3}\pi a$; so that the volume of any pedal (P) becomes

$$P=\tfrac{4}{3}\pi a(a^2+x^2+y^2+z^2)=\tfrac{4}{3}\pi a(a^2+r^2).$$

When the origin of (P) is without the sphere, the pedal consists, of course, of two distinct sheets, each passing through the origin and touching the primitive; the volume of the pedal, as above given, is the *difference* of the volumes enclosed by these sheets. When the sphere diminishes to a point, the volumes of all pedals vanish according to the formula; so that we must regard the pedal of a point as consisting of *two* coincident spherical sheets.

In like manner the pedal of a tubular surface would, in general, consist of distinct sheets which would coincide when the primitive degenerated to a line. Although the pedal surface, therefore, still exists when two dimensions of the primitive are supposed to vanish—being, in fact, still the envelope of spheres whose diameters are the radii vectores of the curve—its volume must be regarded as evanescent.

The case is otherwise, however, when one only of the three dimensions of the primitive is supposed to vanish. Such a surface (S′) would consist of two coincident sheets, and would, therefore, enclose no space; to the eye, in fact, it would not be distinguishable from some definite portion of an ordinary surface. Its pedal, however, would be of a compound nature—consisting, *first*, of a surface (P′) of the same nature as (S′), undistinguishable to the eye from a portion of its ordinary pedal, and, *secondly*, of the simple pedal (P) of the curve (C) forming the contour of the primitive (S′). The *volume* of the compound pedal, however, would be simply that of the pedal (P) of the contour (C). This volume, therefore, properly interpreted, ought to be deducible from our general formulæ.

It must be observed, however, that although the *form* of the pedal of a curve (C) is invariable, its volume must be differently estimated according to the nature of the two-dimensional surface (S′) of which the curve is supposed to form the contour. To render this more evident, it will be convenient to regard the pedal of a curve, not only as the envelope of a sphere, but also as the locus of a circle whose magnitude varies at the same time that its plane rotates about a fixed point, the pedal-origin. This circle, in fact, is the *characteristic* of the pedal; its plane is perpendicular to the tangent at a point M on the curve (C), and its chords, though the origin, are the perpendiculars upon the *several* tangent planes of (S′) at the point M of its contour.

Remembering now the convention of art. 8 with respect to the signs of these perpendiculars, and the relation between the same and those of the corresponding volume-elements, we easily conclude that the volume of the pedal (P) will be the difference of the volumes of the surfaces generated by the segments into which the characteristic circle is divided by the perpendicular p' upon the *ordinary* tangent plane of (S′) at the point M of its contour.

The most interesting case, and the only one we shall examine further, is when the surface (S′) coincides with the developable of which (C) is the cuspidal edge. The perpendicular p' then coincides with that let fall on the osculating plane of the primitive curve (C); through it pass the planes of two consecutive characteristics, and the locus of its extremity is the cuspidal edge of the pedal (P), and at the same time the curve to which, as is well known, the pedal surface of the developable (S′) resolves itself. The volume of the pedal (P) has now the simplest possible definition, and the double integrals of art. 8, by means of which this volume may be expressed, are easily reducible to single ones.

12. To effect this reduction, we will first express any perpendicular p_0 by means of p' the perpendicular on the osculating plane of (C)—parallel therefore to the binormal—and

p the perpendicular on the rectifying plane; the latter will of course be at right angles to p' and parallel to the principal normal. Since p' and p are perpendicular chords of a circle, passing through the same point of its circumference, we have at once

$$p_0 = p' \cos\phi + p \sin\phi,$$

where ϕ is the angle between p' and p_0.

Further, the direction-cosines of p_0, that is to say $\cos\alpha$, $\cos\beta$, $\cos\gamma$, may in like manner be expressed by means of those of p' and p, which we will denote respectively by λ', μ', ν' and λ, μ, ν. For the projections on p_0, p', p of the linear unit, set off on any line through the origin, are clearly, again, chords of a circle, so that, operating successively on the three coordinate axes, we readily deduce the relations

$$\cos\alpha = \lambda' \cos\phi + \lambda \sin\phi,$$
$$\cos\beta = \mu' \cos\phi + \mu \sin\phi,$$
$$\cos\gamma = \nu' \cos\phi + \nu \sin\phi.$$

Lastly, representing by $d\theta$ the angle between the planes of two consecutive characteristics, in other words the *angle of contact* of the primitive curve (C), the surface-element $d\sigma$ of the unit-sphere will have the value

$$d\sigma = \sin\phi \, d\phi \, d\theta.$$

We have now merely to substitute the above values in the several integrals of art. 8, and to effect the integration according to ϕ between the limits 0 and π, regarding thereby p, p', λ, μ, ν, λ', μ', ν' as constants. This may be readily done; the nineteen results are deducible by appropriate permutations of λ, λ'; μ, μ'; and ν, ν', in accordance with the corresponding suffixes $_1$, $_2$, and $_3$, from the following six expressions:—

$$A_1 = \frac{\pi}{8}\int(3\lambda p^2 + 2\lambda' pp' + \lambda p'^2)d\theta.$$

$$A_{11} = \frac{\pi}{8}\int[(3\lambda^2 + \lambda'^2)p + 2\lambda\lambda' p']d\theta.$$

$$A_{23} = \frac{\pi}{8}\int[(3\mu\nu + \mu'\nu')p + (\mu\nu' + \mu'\nu)p']d\theta.$$

$$A_{111} = \frac{3\pi}{8}\int\lambda(\lambda^2 + \lambda'^2)d\theta.$$

$$A_{112} = \frac{\pi}{8}\int[(3\lambda^2 + \lambda'^2)\mu + 2\lambda\lambda'\mu']d\theta.$$

$$A_{123} = \frac{\pi}{8}\int(3\lambda\mu\nu + \lambda'\mu'\nu + \lambda\mu'\nu' + \lambda'\mu\nu')d\theta.$$

By means of the equations to the curve the nine quantities involved in these integrals are readily expressible as functions of a single variable. This done, the integration in each case is to be extended to all points of the primitive curve (C).

13. I do not enter into the several interesting questions which here suggest them-

selves—as to the nature of the cubic locus of the origins of pedals of the present kind which have a constant volume, the conditions under which this locus degenerates to a quadric, and the position of the origin of the pedal of least volume—but pass at once to the case of a plane primitive curve, every pedal of which will be a surface generated by a circle, through two fixed points, whose magnitude varies at the same time that its plane rotates around the line joining those points. Taking the plane of the primitive as the coordinate plane of xy, we have clearly

$$p'=0,\ \lambda'=\mu'=\nu=0,\ \nu'=1,$$

and consequently

$$A_3=A_{23}=A_{31}=A_{113}=A_{223}=A_{333}=A_{123}=0,$$

$$A_1=\frac{3\pi}{8}\int\lambda p^2 d\theta,\qquad A_2=\frac{3\pi}{8}\int\mu p^2 d\theta,$$

$$A_{11}=\frac{3\pi}{8}\int\lambda^2 p d\theta,\qquad A_{22}=\frac{3\pi}{8}\int\mu^2 p d\theta,$$

$$A_{12}=\frac{3\pi}{8}\int\lambda\mu p d\theta,\qquad A_{33}=\frac{\pi}{8}\int p d\theta,$$

$$A_{111}=\frac{3\pi}{8}\int\lambda^3 d\theta,\qquad A_{222}=\frac{3\pi}{8}\int\mu^3 d\theta,$$

$$A_{112}=\frac{3\pi}{8}\int\lambda^2\mu d\theta,\qquad A_{221}=\frac{3\pi}{8}\int\lambda\mu^2 d\theta,$$

$$A_{331}=\frac{\pi}{8}\int\lambda d\theta,\qquad A_{332}=\frac{\pi}{8}\int\mu d\theta.$$

When the primitive is a plane *closed* curve, the last six integrals, in general, vanish, and the locus of origins of equal pedals again degenerates to a quadric surface. The origin of the least pedal does not generally coincide with the *Krümmungs-Schwerpunct*, since A_1, A_2 have no longer the same values as in art. 4; it coincides with the centre of the primitive, however, whenever the latter possesses such a point. For instance, for a primitive circle (a) it will be found, on taking its centre for origin, that, with the exception of three, all the foregoing integrals vanish, and that these three acquire the values

$$A_{11}=A_{22}=\tfrac{3}{8}\pi^2 a,\quad A_{33}=\tfrac{1}{4}\pi^2 a.$$

The volume of the central or least pedal, (P_0), which is here a surface generated by the rotation of a circle with radius $\frac{a}{2}$ about one of its tangents, is easily found to be $\frac{1}{4}\pi^2 a^3$, so that the volume of any other pedal will, by art. 8, be

$$P=\frac{\pi^2}{8}a(3x^2+3y^2+2z^2+2a^2),$$

and the locus of origins of pedals of the same volume a prolate spheroid.

14. To return to the case of surfaces: I propose to consider next the pedals of the ellipsoid, which, ever since the publication of FRESNEL'S researches on Light, have been

regarded with especial interest. The application to them of the foregoing principles will lead us to several new results.

With a view to this application, and in continuation of the subject of art. 10, I may add that when the primitive surface is symmetrical with respect to three rectangular planes, the integrals A_{12}, A_{23}, A_{31} likewise vanish, on taking these planes of symmetry for coordinate planes. In virtue of this property, which is evident from an inspection of the values in art. 8, the expression for the volume of any pedal assumes the simple form

$$P=P_0+A_{11}x^2+A_{22}y^2+A_{33}z^2.$$

If, further, as in the case of the ellipsoid, *the primitive be a closed convex surface*, the coefficients

$$A_{11}=\int p_0 d\sigma \cos^2\alpha, \quad A_{22}=\int p_0 d\sigma \cos^2\beta, \quad A_{33}=\int p_0 d\sigma \cos^2\gamma$$

will manifestly be sums of elements of the same sign, so that *the locus* (A) *of equal pedal origins will be an ellipsoid* whose axes coincide with the axes of symmetry of the primitive.

15. For the primitive ellipsoid

$$\frac{x^2}{a_1}+\frac{y^2}{a_2}+\frac{z^2}{a_3}=1,$$

the squares of whose semiaxes, written in descending order of magnitude, we will suppose to be a_1, a_2, a_3, we have the well-known formulæ

$$\cos\alpha=\frac{x}{a_1}p_0, \quad \cos\beta=\frac{y}{a_2}p_0, \quad \cos\gamma=\frac{z}{a_3}p_0,$$

$$\frac{1}{p_0^2}=\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}+\frac{z^2}{a_3^2}=\frac{1}{a_1\cos^2\alpha+a_2\cos^2\beta+a_3\cos^2\gamma},$$

$$3P_0=\int p_0^3 d\sigma=\frac{1}{a_1a_2a_3}\int p_0^7 ds.$$

Both these equivalent expressions for the volume of the central or least pedal have their advantages. In the second the integration is supposed to be extended to all points of the ellipsoid; in the first, after expressing α, β, γ and thence p_0 by means of two suitable independent variables, to all points of the unit sphere. The limits in the latter case will not involve the axes, and by partial differentiation we shall clearly have

$$\frac{\partial P_0}{da_1}=\int p_0^2\frac{\partial p_0}{da_1}d\sigma=\tfrac{1}{2}\int d\sigma\, p_0\cos^2\alpha=\frac{A_{11}}{2},$$

with similar formulæ for A_{22} and A_{33}; so that the volume of any pedal whatever will be given by the formula

$$P=P_0+2\frac{\partial P_0}{da_1}x^2+2\frac{\partial P_0}{da_2}y^2+2\frac{\partial P_0}{da_3}z^2;$$

that is to say, it will be obtained by simple differentiation of the expression for P_0. At the same time it will be observed that P_0, being a homogeneous function of a_1, a_2, a_3 of the degree $\frac{3}{2}$, satisfies, identically, the relation

$$3P_0=2a_1\frac{\partial P_0}{da_1}+2a_2\frac{\partial P_0}{da_2}+2a_3\frac{\partial P_0}{da_3},$$

or, retaining the more convenient symbols A_{11}, A_{22}, A_{33},

$$3P_0=a_1A_{11}+a_2A_{22}+a_3A_{33}.$$

16. From this, and the general formula for P in art. 14, a very simple relation may be at once established between the volume of the central pedal and that of any other whose origin is on one of the diagonals of the rectangular parallelopiped circumscribed to the ellipsoid. For the coordinates of any point on such a diagonal are given by the equations

$$\frac{x^2}{a_1}=\frac{y^2}{a_2}=\frac{z^2}{a_3}=\frac{r^2}{a},$$

where r is the radius vector to the point, and $a=a_1+a_2+a_3$ the square of the semi-diagonal in question. On substituting these values the two formulæ for P and P_0 give

$$P=\frac{a+3r^2}{a}P_0.$$

When $r^2=a$, the origin of (P) coincides with a corner of the parallelopiped; and when $3r^2=a$, it is a point on the ellipsoid; so that we may say, *the volume of the pedal whose origin is at any corner of the rectangular parallelopiped circumscribed to the primitive ellipsoid is four times that of the central pedal, and double that of the pedal at any one of the eight points wherein the ellipsoid is pierced by the diagonals of the parallelopiped.*

17. In order to establish further relations we will represent, generally, by x_i, y_i, z_i and r_i the coordinates and radius vector of any point (i) in space, and consider, first, the pedals (P_1), (P_2), (P_3) whose origins are at the extremities $(_1)$, $(_2)$, $(_3)$ of any three conjugate diameters of a quadric (S′) concentric and co-axal with the primitive ellipsoid (S). The squared semiaxes of (S′) being a'_1, a'_2, a'_3 we have, of course,

$$x_1^2+x_2^2+x_3^2=a'_1,$$
$$y_1^2+y_2^2+y_3^2=a'_2,$$
$$z_1^2+z_2^2+z_3^2=a'_3;$$

so that by substituting successively, in the general formula for (P), art. 14, the coordinates of the three points under consideration, and adding together the resulting equations, we have

$$P_1+P_2+P_3=3P_0+a'_1A_{11}+a'_2A_{22}+a'_3A_{33}=3\overline{P}.$$

The pedal $(\overline{P})$, whose volume is here put equal to one-third of the *constant* sum of the other three volumes, is easily seen, by the general formula for P, art. 14, to be that whose origin is at one of the points $\left(\sqrt{\frac{a'_1}{3}}, \sqrt{\frac{a'_2}{3}}, \sqrt{\frac{a'_3}{3}}\right)$, where the quadric (S′) is pierced by the diagonals of its circumscribed parallelopiped. If, then, we agree to take the volume of a pedal positively or negatively according as the diameter upon which its origin lies meets the quadric (S′) in real or imaginary points, we may say that *the algebraical sum of the volumes of three ellipsoid-pedals, whose origins are at the extremities of any conjugate diameters of a concentric and co-axal quadric, is constant, and equal to three times the volume of the pedal at the point where this quadric is pierced by a diagonal of its circumscribed rectangular parallelopiped.*

We may add, too, that the sum of the three pedal-volumes corresponding to origins situated at the extremities of conjugate diameters is not only invariable for one and the same quadric (S′), but for all concentric and co-axal quadrics which are inscribed in rectangular parallelopipeds, themselves inscribed in one and the same locus (A) of equal pedal origins. For the axes of all such quadrics clearly satisfy the condition

$$a'_1A_{11}+a'_2A_{22}+a'_3A_{33}=\text{const.}$$

18. When the quadric (S′) is not only concentric and co-axal with the primitive ellipsoid, but also similar to it, the diagonals of their circumscribed rectangular parallelopipeds coincide in direction; so that by art. 16, and putting

$$3r^2=a'_1+a'_2+a'_3=a',$$

the last relation becomes

$$P_1+P_2+P_3=3\overline{P}=3\frac{a+a'}{a}P_0.$$

When $a'=a$, that is to say, when the quadric (S′) coincides with the primitive ellipsoid, we learn that *the sum of the volumes of the three pedals whose origins are at the extremities of any conjugate diameters of the primitive ellipsoid is constant, and equal to six times the volume of the central or least pedal.*

The three pedals whose origins are the vertices of the primitive ellipsoid are, of course, included in this theorem.

19. When the quadric (S′) is a sphere, the conjugate diameters are at right angles to each other, and the diagonals of the circumscribed parallelopiped (cube) are equally inclined to the axes of the ellipsoid; hence *the sum of the volumes of the ellipsoid-pedals whose origins are the three vertices of any tri-rectangular triangle on a concentric sphere is constant, and equal to three times the volume of the pedal at a point on the sphere equidistant from the axes of the ellipsoid.* The value of this constant sum is

$$3P_0+r^2(A_{11}+A_{22}+A_{33}).$$

20. Lastly, when the quadric (S′) is an ellipsoid confocal with the primitive, we may put

$$a'_1-a_1=a'_2-a_2=a'_3-a_3=k^2,$$

and substitute the values of a'_1, a'_2, a'_3 in the general equation of art. 17. By so doing we find

$$P_1+P_2+P_3=6P_0+k^2(A_{11}+A_{22}+A_{33}).$$

Comparing this, therefore, with the expression at the end of the last article, we learn that *the sum of the volumes of the three pedals whose origins are at the extremities of any conjugate diameters of an ellipsoid confocal with the primitive is equal to double the sum of the volumes of the three pedals at the extremities of any three orthogonal diameters of a concentric sphere the square on whose radius is half the difference of the squares on the like-directed semiaxes of the confocals.* Of this general theorem the one at the end of art. 18 is a particular instance, corresponding to the case where the confocal ellipsoids coincide, and consequently $k=0$.

21. From the fundamental formula, written thus,

$$\mathrm{H}=\frac{\mathrm{P}-\mathrm{P}_0}{r^2}=\mathrm{A}_{11}\cos^2\lambda+\mathrm{A}_{22}\cos^2\mu+\mathrm{A}_{33}\cos^2\nu,$$

we may deduce further relations, as well as a construction for the volume of the pedal at any point. In the first place we learn that the *linear* magnitude H is constant at all points of the same radius vector; and secondly, that it is the limit to which $\frac{\mathrm{P}}{r^2}=h$ approaches as the origin of the pedal recedes from the centre. This line h, being the altitude of a parallelopiped (of the same volume as the pedal) having for its base the square on the radius vector, we propose, for convenience of enunciation, to call the *pedal-altitude* at the point under consideration. Thus H will be the pedal-altitude at infinity on the line (λ, μ, ν); A_{11}, A_{22}, A_{33}, respectively, the pedal-altitudes at infinity on the three axes, and $(\mathrm{A}_{11}+\mathrm{A}_{22}+\mathrm{A}_{33})$ that on the line equally inclined to the three axes.

Imagine now a central ellipsoid-pedal $(\dot{\mathrm{P}})$, concentric and co-axal with the primitive, and such that the squares on its semiaxes are respectively proportional to the altitudes A_{11}, A_{22}, A_{33}. It is plain from the last equation that the squares on its radii vectores will be proportional to the pedal-altitudes at infinity on those vectores. The pedal-altitude at infinity on any line being thus determined by the auxiliary pedal $(\dot{\mathrm{P}})$, that at any other point on the same line is easily found, and thence also the parallelopiped, equal in volume to the pedal which has that point for origin.

22. Between the pedal-altitudes at different points in space numerous relations might be established; we shall limit ourselves to one or two. Let (1), (2), (3) now denote the extremities of any three diameters, at right angles to each other, of the concentric and co-axal quadric (S′) before considered. Then the addition of the three formulæ (similar to the one last written) which refer to these extremities gives

$$h_1+h_2+h_3=\mathrm{P}_0\left(\frac{1}{a'_1}+\frac{1}{a'_2}+\frac{1}{a'_3}\right)+\mathrm{A}_{11}+\mathrm{A}_{22}+\mathrm{A}_{33}=3\bar{h},$$

since by a well-known theorem

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{1}{a'_1}+\frac{1}{a'_2}+\frac{1}{a'_3}.$$

The pedal-altitude $\bar{h}$, which is here put equal to the *constant* arithmetic mean of the other three, corresponds to the point on the quadric (S′) which is equidistant from its three axes, as may be easily seen by putting, in the formula of art. 21,

$$\cos^2\lambda=\cos^2\mu=\cos^2\nu=\tfrac{1}{3},$$

and observing that for such a point

$$\frac{3}{r^2}=\frac{1}{a'_1}+\frac{1}{a'_2}+\frac{1}{a'_3}.$$

Hence *the algebraical sum of the three pedal-altitudes at the extremities of any three orthogonal diameters of a quadric, concentric and co-axal with the primitive ellipsoid, is constant, and equal to three times the pedal-altitude at the extremity of a diameter of the quadric equally inclined to its axes.*

We may add, too, that *this sum is not only invariable for one and the same quadric* (S′), *but for all concentric and co-axal quadrics which pass through one and the same point, equidistant from the principal diametral planes of the primitive ellipsoid.* The quadric (S′) being a sphere, the pedal-altitudes at its several points are, of course, proportional to the pedal-volumes; so that we obtain again the theorem of art. 19.

23. Before proceeding to the actual calculation of the volume of an ellipsoid-pedal, we may remark, lastly, that for any four origins situated on a concentric and co-axal quadric the corresponding pedal-volumes satisfy the relation

$$\begin{vmatrix} P_1, & x_1^2, & y_1^2, & z_1^2 \\ P_2, & x_2^2, & y_2^2, & z_2^2 \\ P_3, & x_3^2, & y_3^2, & z_3^2 \\ P_4, & x_4^2, & y_4^2, & z_4^2 \end{vmatrix} = 0,$$

into the geometrical meaning of which, however, we will, at present, not inquire further.

24. I propose to show, in the next place, that the volume of any pedal may be expressed, symmetrically, by means of the first partial differential coefficients of the definite integral

$$V = \int_0^\infty \frac{dv}{\sqrt{(v+a_1)(v+a_2)(v+a_3)}}.$$

It is well known that when the coordinates x, y, z of any point of a surface are regarded as functions of two independent variables ϕ and v, we have the following equivalent expressions for three times the volume of the pyramid whose vertex is the coordinate origin, and base the surface-element ds enclosed between the curves $\phi = const.$, $v = const.$ and their respective consecutives:

$$p_0 ds = \begin{vmatrix} x, & y, & z \\ \frac{\partial x}{d\phi}, & \frac{\partial y}{d\phi}, & \frac{\partial z}{d\phi} \\ \frac{\partial x}{dv}, & \frac{\partial y}{dv}, & \frac{\partial z}{dv} \end{vmatrix} dv\, d\phi.$$

25. Now the equation of the primitive ellipsoid will obviously be satisfied, identically, by the assumptions

$$x^2 = a_1 \frac{v}{v+a_3} \cos^2\phi,$$

$$y^2 = a_2 \frac{v}{v+a_3} \sin^2\phi,$$

$$z^2 = a_3 \frac{a_3}{v+a_3};$$

which, when substituted in the above determinant and in the expression for p_0 given in art. 15, lead at once to the expressions

$$p_0 ds = -\frac{a_3\sqrt{a_1a_2}}{2} \cdot \frac{dv\, d\phi}{(v+a_3)^{\frac{3}{2}}}.$$

$$\frac{1}{p_0^2} = \left[\frac{v+a_1}{a_1}\cos^2\phi + \frac{v+a_2}{a_2}\sin^2\phi\right]\frac{1}{v+a_3}.$$

Substituting these values in the second expression for P_0, given in art. 15, extending the integration over the ellipsoid-octant,—whereby the limits of ϕ will clearly be 0 and $\frac{\pi}{2}$, whilst those of v will be 0 and ∞,—and taking eight times the result, we have

$$3P_0=-\frac{4}{\sqrt{a_1a_2}}\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{(v+a_3)^{\frac{3}{2}}dv.d\phi}{\left[\frac{v+a_1}{a_1}\cos^2\phi+\frac{v+a_2}{a_2}\sin^2\phi\right]^3},$$

whence, by differentiation, we deduce

$$A_{33}=2\frac{\partial P_0}{da_3}=-\frac{4}{\sqrt{a_1a_2}}\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{(v+a_3)^{\frac{1}{2}}dv\,d\phi}{\left[\frac{v+a_1}{a_1}\cos^2\phi+\frac{v+a_2}{a_2}\sin^2\phi\right]^3}.$$

The integration according to ϕ presents no difficulty, and when effected gives the result

$$A_{33}=\frac{\pi}{4}\int_0^\infty\left[\frac{3a_1^2}{(v+a_1)^2}+\frac{2a_1a_2}{(v+a_1)(v+a_2)}+\frac{3a_2^2}{(v+a_2)^2}\right]\frac{(v+a_3)dv}{\sqrt{R}},$$

where, for brevity, we have put

$$R=(v+a_1)(v+a_2)(v+a_3).$$

26. A more convenient form can be given to the above expression for A_{33} by introducing the partial differential coefficients of the two symmetrical integrals

$$V=\int_0^\infty\frac{dv}{\sqrt{R}},\qquad W=\int_0^\infty\frac{vdv}{\sqrt{R}}.$$

In fact if, for brevity, we indicate the results of the operations

$$\frac{\partial}{da_1},\ \frac{\partial}{da_2}\ \ldots.\ \frac{\partial^2}{da_1^2},\ \frac{\partial^2}{da_1da_2},\ \ldots.$$

performed on any subject, by giving to the symbol of that subject the suffixes $_1, _2, \ldots. _{11}, _{12}, \ldots$, we shall have

$$\frac{1}{\pi}A_{33}=a_3(a_1^2V_{11}+2a_1a_2V_{12}+a_2^2V_{22})+a_1^2W_{11}+2a_1a_2W_{12}+a_2^2W_{22},$$

as may be easily verified.

27. This expression, however, may itself be resolved into a simpler one involving V_1, V_2, V_3 alone. To effect this resolution we may observe that, in virtue of the identity

$$\frac{dR}{dv}=R_1+R_2+R_3,$$

we have

$$W_1+W_2+W_3=-\tfrac{1}{2}\int_0^\infty(R_1+R_2+R_3)\frac{vdv}{\sqrt{R^3}}=\int_0^\infty vd\left(\frac{1}{\sqrt{R}}\right);$$

from which, by partial integration, we deduce

$$W_1+W_2+W_3=-V,$$

since $\frac{v}{\sqrt{R}}$ clearly vanishes at both limits. From this expression, again, we obtain by differentiation the relations

$$W_{11}+W_{12}+W_{13}=-V_1,$$
$$W_{12}+W_{22}+W_{23}=-V_2,$$
$$W_{13}+W_{23}+W_{33}=-V_3.$$

By subjecting the integral V to a precisely similar treatment, it will be found that

$$\begin{aligned} V_1+V_2+V_3 &= -\frac{1}{\sqrt{a_1a_2a_3}}, \\ &= -2a_1(V_{11}+V_{12}+V_{13}), \\ &= -2a_2(V_{12}+V_{22}+V_{23}), \\ &= -2a_3(V_{13}+V_{23}+V_{33}). \end{aligned}$$

Further, since

$$V_1 = -\tfrac{1}{2}\int_0^\infty \frac{1}{v+a_1}\cdot\frac{dv}{\sqrt{R}}, \text{ \&c. } \ldots$$

$$V_{12}=\tfrac{1}{4}\int_0^\infty \frac{1}{(v+a_1)(v+a_2)}\,\frac{dv}{\sqrt{R}}, \text{ \&c. } \ldots$$

we have, on resolving the coefficients of $\frac{dv}{\sqrt{R}}$ in V_{12}, V_{23}, V_{31} into partial fractions,

$$2(a_1-a_2)V_{12}=V_1-V_2,$$
$$2(a_2-a_3)V_{23}=V_2-V_3,$$
$$2(a_3-a_1)V_{31}=V_3-V_1;$$

and in like manner we also find that

$$2(a_1-a_2)W_{12}=a_2V_2-a_1V_1,$$
$$2(a_2-a_3)W_{23}=a_3V_3-a_2V_2,$$
$$2(a_3-a_1)W_{31}=a_1V_1-a_3V_3.$$

28. Now the last four groups of equations clearly suffice for the expression of A_{33} (art. 26) in terms of V_1, V_2, V_3, and thence, by mere permutations of suffixes, we may obtain the values of A_{11}, A_{22}. The results, after due simplification, may be thus written:

$$A_{11}=-\frac{\pi}{2}\Big[(a_2+a_3)a_1V_1+(a_3+2a_1)a_2V_2+(2a_1+a_2)a_3V_3\Big],$$

$$A_{22}=-\frac{\pi}{2}\Big[(2a_2+a_3)a_1V_1+(a_3+a_1)a_2V_2+(a_1+2a_2)a_3V_3\Big],$$

$$A_{33}=-\frac{\pi}{2}\Big[(a_2+2a_3)a_1V_1+(2a_3+a_1)a_2V_2+(a_1+a_2)a_3V_3\Big].$$

From these values of the pedal-altitudes at infinity on each of the axes (art. 21) we obtain, by addition, the following value of the pedal-altitude at infinity on a line equally inclined to these axes:

$$A_{11}+A_{22}+A_{33}=-2\pi[(a_2+a_3)a_1V_1+(a_3+a_1)a_2V_2+(a_1+a_2)a_3V_3].$$

Again, in virtue of the relation at the end of art. 15, we at once deduce the following expression for the volume of the central pedal of the ellipsoid,

$$P_0=-\frac{\pi}{2}\left[m_1a_1V_1+m_2a_2V_2+m_3a_3V_3\right];$$

if, for brevity, we put

$$3m_1=(a_1+a_2+a_3)(a_2+a_3)+a_2^2+a_3^2,$$

$$3m_2=(a_1+a_2+a_3)(a_3+a_1)+a_3^2+a_1^2,$$

$$3m_3=(a_1+a_2+a_3)(a_1+a_2)+a_1^2+a_2^2.$$

Lastly, for the volume of any pedal (P) whose origin A is at x, y, z, we have the expression

$$P=-\frac{\pi}{2}\left[M_1.a_1V_1+M_2.a_2V_2+M_3.a_3V_3\right];$$

where again, for brevity, we put

$$3M_1=(3r^2+a)(a_2+a_3)+3(a_2y^2+a_3z^2)+a_2^2+a_3^2,$$

$$3M_2=(3r^2+a)(a_3+a_1)+3(a_3z^2+a_1x^2)+a_3^2+a_1^2,$$

$$3M_3=(3r^2+a)(a_1+a_2)+3(a_1x^2+a_2y^2)+a_1^2+a_2^2,$$

r^2 and a being, as usual, abbreviations for $x^2+y^2+z^2$ and $a_1+a_2+a_3$. The volume of the primitive ellipsoid, when expressed by means of V_1, V_2, V_3, is

$$S=-\frac{4\pi}{3}\left[a_2a_3.a_1V_1+a_3a_1.a_2V_2+a_1a_2.a_3V_3\right],$$

as is at once evident from one of the relations in art. 27. The integral V itself, when thus expressed, has the value

$$V=-2[a_1V_1+a_2V_2+a_3V_3];$$

for it may readily be shown to be a homogeneous function of a_1, a_2, a_3 of the degree $-\frac{1}{2}$.

I do not dwell upon the many interesting expressions of S and V by means of pedal-volumes, but proceed at once to the expression of the foregoing results by means of elliptic integrals.

29. The integral V, by means of whose partial differential coefficients the volumes of all pedals have been expressed, is at once converted into an elliptic integral of the first kind by the substitution

$$\sin^2\varphi=\frac{a_1-a_3}{v+a_1},$$

whereby the limits 0 and ∞ of v will correspond, respectively, to the limits θ and 0 of ϕ, provided

$$\theta=\cos^{-1}\sqrt{\frac{a_3}{a_1}}=\sin^{-1}\sqrt{\frac{a_1-a_3}{a_1}}.$$

The result of this substitution is easily found to be

$$V=\frac{2}{\sqrt{a_1-a_3}}\int_0^\theta\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}=2\frac{F}{\sqrt{a_1-a_3}},$$

where

$$k^2=\frac{a_1-a_2}{a_1-a_3}$$

is clearly positive and less than unity.

Representing also, with LEGENDRE, by E the elliptic integral, of the second kind,

$$E(\theta,k)=\int_0^\theta d\phi\sqrt{1-k^2\sin^2\phi},$$

and differentiating the preceding value of V, it will be found that

$$V_1=\qquad -\frac{1}{a_1-a_2}\cdot\frac{F}{\sqrt{a_1-a_3}}+\frac{1}{a_1-a_2}\cdot\frac{E}{\sqrt{a_1-a_3}},$$

$$V_2=\frac{a_3}{a_2-a_3}\frac{1}{\sqrt{a_1a_2a_3}}+\frac{1}{a_1-a_2}\cdot\frac{F}{\sqrt{a_1-a_3}}-\frac{a_1-a_3}{(a_1-a_2)(a_2-a_3)}\cdot\frac{E}{\sqrt{a_1-a_3}},$$

$$V_3=\frac{-a_2}{a_2-a_3}\frac{1}{\sqrt{a_1a_2a_3}}\qquad +\frac{1}{a_2-a_3}\cdot\frac{E}{\sqrt{a_1-a_3}}.$$

By substituting these values in the formulæ of art. 20, we might at once obtain the values of A_{11}, A_{22}, A_{33}, P_0, and P expressed in elliptic integrals. Since the volume of any pedal (P), however, may be deduced from that of the central pedal (P_0) by mere differentiation (art. 15), the following complete expression for P_0 will here suffice:—

$$P_0=\frac{\pi}{6}\left[(2a-a_1)\sqrt{\frac{a_2a_3}{a_1}}+(aa_3+a_3^2-a_1a_2)\frac{F}{\sqrt{a_1-a_3}}+2(a_1-a_3)a\frac{E}{\sqrt{a_1-a_3}}\right].$$

This expression, I may add, agrees precisely with the one first obtained by Professor TORTOLINI in 1844*.

30. If we allow a_3 to diminish indefinitely, the amplitude θ approaches the limit $\frac{\pi}{2}$, and the modulus k acquires the value

$$k_1=\sqrt{\frac{a_1-a_2}{a_1}}.$$

* CRELLE's Journal, vol. xxxi. p. 28. At the time the present paper was communicated to the Royal Society I was under the impression that the central pedal of the ellipsoid was the only one whose volume had hitherto been calculated. I have since found that Dr. MAGENER, in a paper "On the Cubature of Ellipsoid-pedals" (GRUNERT's Journal, t. xxxiv. 1860), first gave the complete expression for P in elliptic integrals. Although the simple relation between P_0 and P, above referred to, appears to have escaped Dr. MAGENER's notice, it is due to him to state that he not only determined the loci (A) of the origins of ellipsoid-pedals of equal volume, but also succeeded in giving to P a very interesting and symmetrical form, by introducing the partial differential coefficients of the well-known double integral to which JACOBI, in 1833 (CRELLE's Journal, vol. x.), reduced the quadrature of the reciprocal of the primitive ellipsoid.

The elliptic functions E and F thus become transformed into the complete ones $E\left(\frac{\pi}{2}, k_1\right)$ and $F\left(\frac{\pi}{2}, k_1\right)$, or more simply, E_1 and F_1.

Representing generally by [U] the limit to which any function U approaches when a_3 diminishes indefinitely, we deduce from the expressions in art. 29 the limiting values

$$[V_1]=\frac{1}{a_1-a_2}\cdot\frac{E_1}{\sqrt{a_1}}-\frac{1}{a_1-a_2}\cdot\frac{F_1}{\sqrt{a_1}},$$

$$[V_2]=-\frac{a_1}{a_2}\cdot\frac{1}{a_1-a_2}\cdot\frac{E_1}{\sqrt{a_1}}+\frac{1}{a_1-a_2}\cdot\frac{F_1}{\sqrt{a_1}},$$

$$[V_3]=\infty,\quad [V_3\sqrt{a_3}]=-\frac{1}{\sqrt{a_1a_2}},\quad [a_3V_3]=0.$$

$$[P_0]=\frac{\pi}{6}\sqrt{a_1}\{2(a_1+a_2)E_1-a_2F_1\}.$$

This last is the volume of the central pedal surface of an ellipse (art. 13). By substitution in art. 28, it will be found that the volume of any other pedal of this curve is given by the formula

$$[P]-[P_0]=\frac{\pi}{2}\frac{\sqrt{a_1}}{a_1-a_2}\Big\{[(2a_1-a_2)E_1-a_2F_1]x^2+[(a_1-2a_2)E_1+a_2F_1]y^2+(a_1-a_2)E_1z^2\Big\},$$

to which expression we should have been led at once had we sought, directly, the values of A_{11}, A_{22}, A_{33} as exhibited in art. 13. In fact when $a_1=a_2$, the above formula may be easily reduced to the one already found in art. 13 for the volume of the pedal surface of a circle.

31. I give, lastly, the modifications of the preceding formulæ which correspond to the special cases of ellipsoids of rotation.

For the prolate spheroid $a_2=a_3$, and

$$V=\frac{2}{\sqrt{a_1-a_3}}\log\left[\sqrt{\frac{a_1-a_3}{a_3}}+\sqrt{\frac{a_1}{a_3}}\right],$$

$$P_0=\frac{\pi}{6}\left\{(2a_1+3a_3)\sqrt{a_1}+\frac{3a_3^2}{\sqrt{a_1-a_3}}\log\left[\sqrt{\frac{a_1-a_3}{a_3}}+\sqrt{\frac{a_1}{a_3}}\right]\right\},$$

$$P=P_0+2\frac{\partial P_0}{da_1}x^2+\frac{\partial P_0}{da_3}(y^2+z^2).$$

At either focus $x^2=a_1-a_3$, $y=z=0$, and the volume of (P) becomes

$$P=P_0+2\frac{\partial P_0}{da_1}(a_1-a_3)=\frac{4\pi}{3}a_1\sqrt{a_1},$$

which is, of course, the volume of the sphere whose diameter is the major axis of the generating ellipse.

For the oblate spheroid $a_1=a_2$, and hence

$$V=\frac{2}{\sqrt{a_1-a_3}}\cos^{-1}\sqrt{\frac{a_3}{a_1}},$$

$$P_0=\frac{\pi}{6}\left[(3a_1+2a_3)\sqrt{a_3}+\frac{3a_1^2}{\sqrt{a_1-a_3}}\cos^{-1}\sqrt{\frac{a_3}{a_1}}\right],$$

$$P=P_0+\frac{\partial P_0}{da_1}(x^2+y^2)+2\frac{\partial P_0}{da_3}z^2;$$

which last formula, when $a_3=0$, is also reducible to the last formula in art. 13 for the volume of any pedal (P) of a circle, regarded as the limit of a surface, one of whose dimensions has been allowed to diminish indefinitely.